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CONTOUR INVARIANTS IN THE THEORY OF FRACTURE OF THERMOELASTIC BODIES*

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Conditions of crack growth in a medium described by the equations of coupled thermoelasticity are studied. Two integrals are written for the contour enclosing the stationary field near the crack tip. The first of these integrals corresponds to the free energy balance and includes the mechanical work. In the limiting cases of the rapid and slow (in thermal sense) cracks it becomes the known J-invariant from which thermal fluxes are excluded. The second integral is formulated as the entropy (heat) flux balance.

The Gibbs thermodynamic relation for a disintegrating body and expression for free energy /1/ together yield the relation connecting the variations in independent parameters l, T and Δ (l is the crack length, T is body temperature and Δ is the displacement caused by the external load P). We can also obtain the expressions for the thermodynamic forces

$$P=\frac{\partial\Phi_v}{\partial\Delta},\quad G\equiv-\frac{\partial\Phi_v}{\partial l}=2\gamma(T),\quad S+2l\frac{\partial\gamma}{\partial T}=-\frac{\partial\Phi_v}{\partial T}$$

given in terms of the free energy Φ_v of the body with a crack. Here S is the total entropy of the body and $\gamma(T)$ is the effective surface energy. The crack in the body grows, if the Irwin force G reaches a critical value depending on the temperature. Such an analysis however is insufficient in a typical situation of nonuniform distribution of the temperature throughout the body.

1. We consider the quasi-equilibrium processes in a thermoelastic body described by the energy balance and the mechanical equilibrium equations

$$\frac{\partial\varepsilon}{\partial t}=\sigma_{ij}\frac{\partial e_{ij}}{\partial t}-\frac{\partial q_j}{\partial x_j},\quad \frac{\partial\sigma_{ij}}{\partial x_j}=0;\quad e_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \tag{1.1}$$

Here ε is the specific internal energy, σ_{ij} are the stresses, e_{ij} the deformations and q_j denote the heat fluxes. If the crack in the body grows at the rate $l_k^{\cdot}$ (plane problem) then introduction of the associated moving coordinate system $y_k=x_k-l_k^{\cdot}t$ is expedient. Then, integration of the first equation of (1.1) over the area $S_{\beta-\alpha}$ bounded by two consecutive contours Γ_α, Γ_β about the crack tip leads, in the absence of the mass, impulse and energy flows across its edge, lead to the equation

$$\int_{\Gamma_\beta}\varkappa_j n_j\,d\Gamma=\int_{\Gamma_\alpha}\varkappa_j n_j\,d\Gamma,\qquad \varkappa_j=\varepsilon l_k^{\cdot}\delta_{kj}-\sigma_{ij}\frac{\partial u_i}{\partial y_k}l_k^{\cdot}-q_j \tag{1.2}$$

provided that the fields of the variables in question are stationary within $S_{\beta-\alpha}$. The integrals (1.2) are invariants, and this means that the total energy fluxes through any contour Γ_β are equal to each other. If no energy enters the body through the crack tip and the fracture point is referred to the body, then the following condition holds for any contour Γ_β /2/:

$$\int_{\Gamma_\beta}\varkappa_j n_j\,d\Gamma=0$$

Let us now consider the difference /2/ between the energy of a particle lying inside the body $(\varepsilon=\varepsilon_v)$ and a particle on the fracture surface $(\varepsilon=\varepsilon_s)$. We have

$$\int_{\Gamma_\beta}\varkappa_j n_j\,d\Gamma=\int_{\Gamma_\beta}l_k^{\cdot}(\varepsilon_v-\varepsilon)\,\delta_{kj}n_j\,d\Gamma=2\gamma_\varepsilon(T_\beta)\,l_k^{\cdot} \tag{1.3}$$

Indeed, the particles with energy ε_s form a thin layer $(-a\leqslant y_l\leqslant a)$ along the crack edge, and the right-hand side of the balance (1.3) is transformed as follows:

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$$\int_{\Gamma_\beta} (\varepsilon_v - \varepsilon)\, l_k^{\cdot} \delta_{kj} n_j \, d\Gamma = l_k^{\cdot} \int_{-a}^{a} (\varepsilon_v - \varepsilon)\, dy_i = 2a\,[\varepsilon(T_\beta)]\, l_k^{\cdot} = 2\gamma_\varepsilon(T_\beta)\, l_k^{\cdot} \tag{1.4}$$

$$(i \neq j;\ [\varepsilon(T)] = \varepsilon_v - \varepsilon_s)$$

Here $\gamma_\varepsilon(T)$ denotes the excess energy corresponding to the point of intersection of the contour Γ_β with the crack surface. When $T_\beta \neq T_\alpha$, the contour integral (1.4) is reduced not to a constant, but to a function of T_β along the crack.

2. To analyze the characteristic temperature field we use the heat flux equation, i.e. the entropy balance s representing a function which can only grow locally

$$\frac{\partial s}{\partial t} = -\frac{\partial h_j}{\partial x_j} + \Pi, \quad h_j = \frac{q_j}{T}, \quad \Pi = \frac{\sigma_{ij}}{T} \frac{\partial e_{ij}^p}{\partial t} + q_j \frac{\partial}{\partial x_j}\left(\frac{1}{T}\right) \tag{2.1}$$

Here h_j denotes the flux, and Π the work done by entropy. In the steady moving region $S_{\beta-\alpha}$ the equation (2.1) simplifies to

$$l_k^{\cdot} \delta_{kj} \frac{\partial s}{\partial y_j} = \frac{\partial h_j}{\partial y_j} + \Pi \tag{2.2}$$

and its integration over $S_{\beta-\alpha}$ yields an equation which shows that the integral over the contour Γ_β is not, in general, an invariant quantity for the entropy fluxes. It is only when

$$\int_{S_{\beta-\alpha}} \Pi \, dS = 0 \tag{2.3}$$

that the above equation reduces to condition of invariance

$$\int_{\Gamma_\alpha} (l_k^{\cdot} \delta_{kj} s - h_j)\, n_j \, d\Gamma = -\int_{S_\alpha} \Pi \, dS = -\frac{l_k^{\cdot}}{T_\alpha} 2\gamma_*(T_\alpha) \tag{2.4}$$

(the value of the constant is obtained from the entropy balance for the region S_α within the contour Γ_α which includes the fracture point). Here we assume that the entropy increases near the crack tip, the instance caused by dissipation of energy on irreversible deformations (the body is nearly elastic /2/). Introduction of the effective surface energy $\gamma_*(T_\alpha)$ implies that the plastic Irwin-Orowan particle is autonomous at the crack tip. The quantity T_α denotes the temperature averaged over the area. If condition (2.3) does not hold, then the contour integrals must be formulated for the velocity fields and for amounts of work done /3/. Let us further take into account the difference in entropy of the particles within the body and at its surface. Then we have

$$\int_{\Gamma_\beta} (l_k^{\cdot} \delta_{kj} s_v - h_j)\, n_j \, d\Gamma = \int_{\Gamma_\beta} l_k^{\cdot} \delta_{kj} (s_v - s)\, n_j \, d\Gamma - \frac{l_k^{\cdot}}{T_\alpha} 2\gamma_*(T_\alpha) \equiv \tag{2.5}$$

$$2\gamma_s(T_\beta) \frac{l_k^{\cdot}}{T_\beta} - 2\gamma_*(T_\alpha) \frac{l_k^{\cdot}}{T_\alpha}; \quad \frac{\gamma_s}{T_\beta} = a\,[s(T_\beta)];$$

$$[s(T)] = s_v(T) - s_s(T).$$

3. Following /4/, we make use of the transformation

$$\varepsilon_v - T_0 s_v = f_v - (T_0 - T)\, s_v, \quad df_v = \sigma_{ij} de_{ij} - s dT \tag{3.1}$$

and write the free energy $f_v = \varepsilon_v - T s_v$ in the form of a series

$$f_v(T_0, e_{ij}) = f_v(T, e_{ij}) - \theta \frac{\partial f_v}{\partial T} + \frac{1}{2} \theta^2 \left(\frac{\partial^2 f_v}{\partial T^2}\right)_*$$

where the last term is taken at $T_* = T - \zeta\theta,\ \theta = T - T_0,\ 0 \leqslant \zeta \leqslant 1$. Since

$$s_v = -\frac{\partial f_v}{\partial T}, \quad \frac{c_e^*}{T_*} = -\left(\frac{\partial^2 f_v}{\partial T^2}\right)_*$$

equation (3.1) becomes

$$\varepsilon_v - T_0 s_v = f_v(T_0, e_{ij}) - \frac{1}{2} \frac{c_e^*}{T_*} \theta^2 \tag{3.2}$$

Next we multiply the balance of the entropy fluxes (2.5) by the reference temperature T_0 and subtract the result from the energy balance (1.3). Further, using the relation (3.2), we obtain

$$\int_{\Gamma_\beta} \Big\{ f_v(T_0, e_{ij})\, l_k^{\cdot} \delta_{kj} - \frac{c_e^*}{2} \frac{\theta^2}{T_0} l_k^{\cdot} \delta_{kj} - \sigma_{ij} \frac{\partial u_i}{\partial y_k} l_k^{\cdot} - \Big(1 - \frac{T_0}{T}\Big) q_j \Big\} n_j\, d\Gamma = 2\gamma_\varepsilon(T_\beta)\, l_k^{\cdot} \Big\{ 1 - \frac{T_0}{T_\beta} \frac{\gamma_s(T_\beta)}{\gamma_\varepsilon(T_\beta)} \Big\} + 2\frac{T_0}{T_\alpha} \gamma_*(T_\alpha)\, l_k^{\cdot} \tag{3.3}$$

In the isothermal case ($T = T_0$) the integral (3.3) transforms directly into the known J-invariant /1,5/ which does not contain the heat fluxes q_j. The Ericksen representation (3.2) can be used to transform the right-hand side of (3.3)

$$\gamma_\varepsilon(T_\beta) - \frac{T_0}{T_\beta} \gamma_s(T_\beta) = a([\varepsilon] - T_0[s]) = a[f(T_0)] - \frac{a}{2} \frac{[c_e^*]}{T_*} \frac{\theta_\beta^2}{T_\beta} = \gamma_f(T_0) - \frac{a}{2} \frac{[c_e^*]}{T_*} \frac{\theta_\beta^2}{T_\beta} = \gamma_0(T_\beta) \tag{3.4}$$

Here γ_0 denotes the Griffith surface energy which is a function of the temperature T_β of the surface layer, determined by two material constants, $\gamma_f(T_0)$ and $b = a[c_e^*]/T_*$.

4. In the linear approximation the free energy f_v, entropy s_v and rheological couplings in the thermoelastic body /6/ have the form

$$f_v(T, e_{ij}) = {}^1\!/_2(\lambda e_{ii} e_{jj} + 2\mu e_{ij} e_{ij}) - 3K\eta\theta e - \frac{c_e}{2} \frac{\theta^2}{T_0} = f_v(T_0, e_{ij}) - 3K\eta\theta e - \frac{c_e}{2} \frac{\theta^2}{T_0}, \quad s_v = 3K\eta e + c_e \frac{\theta}{T_0}, \tag{4.1}$$

$$q_j = -k \frac{\partial \theta}{\partial x_j}, \quad \sigma_{ij} = \lambda e_{ij} \delta_{ij} + 2\mu e_{ij} - 3K\theta\delta_{ij}, \quad K = \lambda + 2\mu/3.$$

where λ, μ are the Lamé coefficients, η is the thermoelastic expansion coefficient and $k = \varkappa c_e$ is the heat conductivity coefficient. In this case the integral (3.3) for the free energy assumes a simpler form

$$\int_{\Gamma_\beta} \Big\{ f_v(T_0, e_{ij})\, \delta_{1j} - \sigma_{ij} \frac{\partial u_i}{\partial y_1} - \frac{c_e}{2} \frac{\theta^2}{T_0} \Big(\delta_{1j} - \frac{2\varkappa}{\theta l^{\cdot}} \frac{\partial \theta}{\partial y_j} \Big) \Big\} \times n_j\, d\Gamma = 2\gamma_0(\theta_\beta) - 2\gamma_*(\theta_\alpha); \quad l_k^{\cdot} = 0, \quad k \neq 1; \quad l_1^{\cdot} = l^{\cdot} \tag{4.2}$$

The contour integral (4.2) replaces the relations which were used in the actual computations /7/ and included a surface integral over $S_{\beta-\alpha}$ (e.g. of the product of volume deformation and the temperature gradient /8/). The second contour integral (for the entropy) is necessary for determining which part of mechanical work expended on fracture is converted into heat. Let Q be the effective amount of heat generated when the length of the crack is increased by one unit of measurement. Then

$$\int_{\Gamma_\beta} \Big\{ c_e\theta \Big(\delta_{1j} + \frac{\varkappa}{\theta l^{\cdot}} \frac{\partial \theta}{\partial y_j} \Big) - 3K\eta T_0 e \delta_{1j} \Big\} n_j\, d\Gamma + Q = 0 \tag{4.3}$$

$$Q = 2\gamma_*(\theta_\alpha) - 2\gamma_s(\theta_\beta)$$

The problem of a crack in a thermoelastic body was solved in /6/ usung the method of asymptotic expansion, and it was assumed that all mechanical work done on fracture G is converted into heat. The quantity G was found as the value of the J-integral /1,5/ after substituting into it the free energy $f_v(T, e_{ij})$ at $T \neq T_0$

Let us consider in this connection the conditions of existence of the J-integral which does not contain the heat fluxes and is therefore invariant with respect to the rate of crack growth. The very first conditions are those of adiabaticity within the countour Γ_β, i.e.

$$\frac{\varkappa}{|\theta l^{\cdot}|} \left| \frac{\partial \theta}{\partial y_j} \right| \sim \frac{\varkappa}{L l^{\cdot}} \ll 1, \quad \frac{\partial \theta}{\partial y} \sim \frac{\theta}{L} \ll \frac{\theta l^{\cdot}}{\varkappa}$$

which hold for the "rapid" (in the thermal sense) cracks. Here L is the linear dimension of the region $S_{\beta-\alpha}$. In this case the integral (4.3) assumes the form

$$\int_{\Gamma_\beta} T_0 s_v \delta_{1j} n_j\, d\Gamma + Q = 0 \tag{4.4}$$

and the following transformations can be carried out in the integrand of (4.2):

$$f_v(T_0, e_{ij}) - \frac{c_e}{2}\frac{\theta^2}{T_0} - \sigma_{ij}\frac{\partial u_i}{\partial y_1} n_j = f_v(T, e_{ij}) + Ts_v -$$
$$T_0 s_v - \sigma_{ij}\frac{\partial u_i}{\partial y_1} n_j = U_v - T_0 s_v - \sigma_{ij}\frac{\partial u_i}{\partial y_1} n_j$$
$$dU_v = \sigma_{ij} de_{ij},\ de_{ij} = de_{ij}{}^e,\ q_j = 0$$

Therefore the integral (4.2) definitely transforms into the known J-invariant

$$J = \int_{\Gamma_\beta} \left(U_v \delta_{1j} - \sigma_{ij}\frac{\partial u_i}{\partial y_1}\right) n_j\, d\Gamma = 2\gamma_q, \quad \gamma_q = \gamma_0(T_\beta) + \gamma_*(T_\alpha) \tag{4.5}$$

Conversely, for the "slow" cracks we have the estimate

$$\frac{\varkappa}{\theta l^\cdot}\left|\frac{\partial\theta}{\partial y_j}\right| \sim \frac{\varkappa}{Ll^\cdot} \gg 1, \quad \frac{\varkappa}{l^\cdot}\frac{\partial\theta}{\partial y} \gg \theta, \quad \frac{\theta k}{Ll^\cdot} \gg 3K\eta T_0 e$$

and this enables us to reduce the integral (4.3) to the form

$$\int_{\Gamma_\beta} \frac{\partial\theta}{\partial y_j} n_j\, d\Gamma + \frac{Q}{k} l^\cdot = 0 \tag{4.6}$$

which corresponds only to the conduction heat fluxes from a point heat source moving with the crack tip. In this isothermal situation of the crack growth the integral (4.2) assumes the form

$$\int_{\Gamma_\beta} \left\{f_v(T_0, e_{ij})\,\delta_{1j} - \sigma_{ij}\frac{\partial u_i}{\partial y_1}\right\} n_j\, d\Gamma = 2\gamma_T, \quad \gamma_T = \gamma_0(T_0) + \gamma_*(T_0) \tag{4.7}$$

As was expected, the work done on fracture in the adiabatic and isothermal case is, generally speaking, different in each case $(\gamma_q \neq \gamma_T)$.

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